

# Project 4

## Part 1)

We have the definition of a Laplace transformation:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

So we have on the left hand side of the equation:

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^{\infty} e^{-st} (af(t) + bg(t)) dt$$

Using integral properties we have:

$$\int_0^{\infty} e^{-st} (af(t) + bg(t)) dt = \int_0^{\infty} e^{-st} af(t) dt + \int_0^{\infty} e^{-st} bg(t) dt = a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt$$

Now on the right hand side we have:

$$a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt$$

We can see that:

$$a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt = a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt$$

and therefore:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

## Part 2b)

We can rewrite:  $\frac{dy}{dt} = y - 4e^{-t}$

as:  $y' - y = -4e^{-t}$ .

Seeing  $-4e^{-t}$ , we know that the particular solution will be in the form  $y_p(t) = Ae^{-t}$ , with first derivative

$$y'_p(t) = -Ae^{-t}$$

We can substitute both  $y_p(t) = Ae^{-t}$  and  $y'_p(t) = -Ae^{-t}$

into:

$$y' - y = -4e^{-t}$$

to get:

$$-Ae^{-t} - Ae^{-t} = -4e^{-t}$$

which we can solve for  $A$  to get  $A = 2$ , which gives us:

$$y_p(t) = 2e^{-t}.$$

Note: we can check that  $y_p(t) = 2e^{-t}$  is a solution to  $y' - y = -4e^{-t}$  by finding its derivative

$y'_p(t) = -2e^{-t}$  and then substituting  $y_p(t)$  and  $y'_p(t)$  into  $y' - y = -4e^{-t}$  to get:

$$-2e^{-t} - 2e^{-t} = -4e^{-t}, \text{ which checks out. So therefore } y_p(t) = 2e^{-t} \text{ is a solution to } y' - y = -4e^{-t}.$$

The familiar homogenous equation of  $y' - y = -4e^{-t}$  is:

$$y' - y = 0$$

which gives us the characteristic equation:

$$r - 1 = 0$$

where we can easily see that  $r=1$ , which give us the complimentary solution:

$$y_c(t) = C_1 e^t, \text{ where } C_1 \text{ is a constant.}$$

Because we get  $y_c(t) = C_1 e^t$  from  $y' - y = 0$ ,  $y_c(t) = 0$  and therefore  $y(t) = y_c(t) + y_p(t)$  is a general solution to  $y' - y = -4e^{-t}$ .

We can rewrite  $y(t) = y_c(t) + y_p(t)$  as:

$$y(t) = C_1 e^t + 2e^{-t}.$$

We are given  $y(0) = 1$  and so we have

$$y(0) = 1 = C_1 e^0 + 2e^{-0} = C_1 + 2.$$

We solve  $C_1 + 2 = 1$  for  $C_1$  to get

$$C_1 = -1$$

,which gives us the general solution:

$$y(t) = -e^t + 2e^{-t}.$$

Note: We can check this by substituting  $y(t)$  and it's first derivative  $y'(t)$  into  $y' - y = -4e^{-t}$ .

### **Part 2c)**

Starting with  $y' - y = -4e^{-t}$  we take the Laplace transform of both sides to get:

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{-4e^{-t}\}$$

$$\Rightarrow \mathcal{L}\{y'\} - \mathcal{L}\{y\} = -4\mathcal{L}\{e^{-t}\}$$

$$\Rightarrow s\mathcal{L}\{y\} - y(0) - \mathcal{L}\{y\} = -4\left(\frac{1}{s+1}\right), \text{ where we get } \frac{1}{s+1} \text{ from the Laplace transform}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \text{ with } a = -1.$$

$$\Rightarrow \mathcal{L}\{y\}(s-1) = -4\left(\frac{1}{s+1}\right) + y(0), \text{ with } y(0) = 1$$

$$\Rightarrow \mathcal{L}\{y\} = -4 \left( \frac{1}{(s+1)(s-1)} \right) + \frac{1}{(s-1)}$$

We now use partial fraction on  $\left( \frac{1}{(s+1)(s-1)} \right)$

$$\frac{1}{(s+1)(s-1)} = \frac{A}{s-1} + \frac{B}{s+1}$$

$$\Rightarrow 1 = A(s+1) + B(s-1)$$

If we let  $s = -1$ , we get  $B = -\frac{1}{2}$ . If we let  $s = 1$  we get  $A = \frac{1}{2}$ , which gives us:

$$\frac{1}{(s+1)(s-1)} = \frac{1}{2} \left[ \left( \frac{1}{s-1} \right) - \left( \frac{1}{s+1} \right) \right]$$

which we substitute into  $\mathcal{L}\{y\} = -4 \left( \frac{1}{(s+1)(s-1)} \right) + \frac{1}{(s-1)}$  to get:

$$\mathcal{L}\{y\} = -4 \frac{1}{2} \left[ \left( \frac{1}{s-1} \right) - \left( \frac{1}{s+1} \right) \right] + \frac{1}{(s-1)}$$

$$\Rightarrow \mathcal{L}\{y\} = -2 \left( \frac{1}{s-1} \right) + 2 \left( \frac{1}{s+1} \right) + \frac{1}{(s-1)}$$

Now we take the inverse Laplace transforms of both sides and using  $\mathcal{L}^{-1} \left\{ \frac{1}{s-a} \right\} = e^{at}$ , to get:

$$\mathcal{L}^{-1} \{ \mathcal{L}\{y\} \} = \mathcal{L}^{-1} \left\{ -2 \left( \frac{1}{s-1} \right) + 2 \left( \frac{1}{s+1} \right) + \frac{1}{(s-1)} \right\}$$

$$\Rightarrow y = \mathcal{L}^{-1} \left\{ -\frac{1}{s-1} + 2 \left( \frac{1}{s+1} \right) \right\}$$

$$\Rightarrow y = \mathcal{L}^{-1} \left\{ -\frac{1}{s-1} \right\} + \mathcal{L}^{-1} \left\{ 2 \left( \frac{1}{s+1} \right) \right\}$$

$$\Rightarrow y = -1 \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} + 2 \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\}$$

$$\Rightarrow y = -e^t + 2e^{-t}, \text{ which was our general solution when we used the other method above.}$$

### **Part 3)**

$x'' + 4x = \sin(3t)$ ,  $x(0) = x'(0) = 0$  is a Spring-Mass system, where there is a mass of 1 unit (implied by the coefficient of  $x''$ ), no damping constant (implied by the lack of  $cx''$  term, and spring constant of 4 (implied by the coefficient of  $x$ ). An outside force of  $f(t) = \sin(3t)$  acts on the system. The initial displacement of 0 is given to us by  $x = 0$ . The initial velocity of 0 is given to us by  $x'(0) = 0$ .

### **Part 4)**

Starting with  $x'' + 4x = \sin(3t)$  we get the characteristic equation  $r^2 + 4 = 0$ , which gives us root  $r = 2i$ , which gives us the complementary equation  $x_c(t) = C_1 \cos(2t) + C_2 \sin(2t)$ .

$\sin(3t)$  suggest to us a particular solution will the form  $x_p(t) = A \sin(3t) + B \cos(3t)$ . We then find the first and second derivatives of  $x_p(t) = A \sin(3t) + B \cos(3t)$  to insert into  $x'' + 4x = \sin(3t)$ , which gives us

$$\begin{aligned} -9A \sin(3t) - 9B \cos(3t) + 4A \sin(3t) + 4B \cos(3t) &= \sin(3t) \\ \Rightarrow -5A \sin(3t) - 5B \cos(3t) &= 1 \sin(3t) + 0 \cos(3t), \end{aligned}$$

We then equate coefficients to get  $-5A = 1$  and  $-5B = 0$   $A = -\frac{1}{5}$ ,  $B = 0$  and the particular solution  $f_p(x) = -\frac{1}{5} \sin(3t)$ .

We form the general solution  $x(t) = x_p + x_c = C_1 \cos(2t) + C_2 \sin(2t) - \frac{1}{5} \sin(3t)$  and take it's derivative with respect to  $t$  and insert the initial conditions  $x'(0) = 0$  into  $x(t)$  and  $x'(t)$  to get:

$$x(0) = 0 = C_1 \cos(2 \cdot 0) + C_2 \sin(2 \cdot 0) - \frac{1}{5} \sin(3 \cdot 0)$$

$$\Rightarrow C_1 = 0$$

and

$$x'(0) = 0 = -2C_1 \sin(2 \cdot 0) + 2C_2 \cos(2 \cdot 0) - \frac{3}{5} \cos(3 \cdot 0)$$

$$\Rightarrow C_2 = \frac{3}{10}$$

And so we have the general solution:

$$x(t) = \frac{3}{10} \sin(2t) - \frac{1}{5} \sin(3t)$$

### **Part 5)**

We take the Laplace transformation of both sides of  $x'' + 4x = \sin(3t)$  to get:

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\sin(3t)\}$$

$$\Rightarrow s\mathcal{L}\{x'\} - x(0) + \mathcal{L}\{4x\} = \frac{3}{s^2 + 9}$$

$$\Rightarrow s[s\mathcal{L}\{x\} - x'(0)] - x(0) + \mathcal{L}\{4x\} = \frac{3}{s^2 + 9}$$

$$\Rightarrow s^2 \mathcal{L}\{x\} + 4 \mathcal{L}\{x\} = \frac{3}{s^2 + 9}, \text{ where } x(0) = x'(0) = 0$$

$$\Rightarrow \mathcal{L}\{x\}(s^2 + 4) = \frac{3}{s^2 + 9}$$

$$\Rightarrow \mathcal{L}\{x\} = \frac{3}{(s^2 + 9)(s^2 + 4)}$$

We now take the inverse laplace of both sides of  $\mathcal{L}\{x\} = \frac{3}{(s^2 + 9)(s^2 + 4)}$  to get

$$\mathcal{L}^{-1}\{\mathcal{L}\{x\}\} = \mathcal{L}^{-1}\left\{\frac{3}{(s^2 + 9)(s^2 + 4)}\right\}$$

$$\Rightarrow x = \mathcal{L}^{-1} \left\{ \frac{3}{(s^2 + 9)(s^2 + 4)} \right\}$$

We will now apply the convolution property:  $x(t) = \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t) = \int_0^t f(\tau)g(t - \tau) \, d\tau$ .

We let  $F(s) = \left( \frac{3}{(s^2 + 9)} \right)$  and  $G(s) = (s^2 + 4)$  and take their inverse Laplace transformations to get

$$f(t) = \sin(3t) \text{ and } g(t) = \frac{1}{2} \sin(2t) \text{ which we plug into } x(t) = \int_0^t f(\tau)g(t - \tau) \, d\tau$$

to get:

$$x(t) = \frac{1}{2} \int_0^t \sin(3\tau) \cdot \sin(2(t - \tau)) \, d\tau, \text{ which we evaluate in Maple to get:}$$

$x(t) = -\frac{1}{5} \sin(3t) + \frac{3}{10} \sin(2t)$ , which we recognize as our solution from when we used a different method above.